

# CHARACTERIZATIONS OF GEOMETRICAL MAPPINGS OF PROJECTIVE SPACES IN TERMS OF THE PROJECTIVE LINEAR GROUP

MARK PANKOV

**ABSTRACT.** Let  $V$  and  $V'$  be vector spaces over division rings. The dimensions of  $V$  and  $V'$  are assumed to be not less than 3 (possibly these vector spaces are infinite-dimensional). Denote by  $\mathcal{P}(V)$  and  $\mathcal{P}(V')$  the projective spaces corresponding to  $V$  and  $V'$ , respectively. Let  $f$  be a PGL-mapping of  $\mathcal{P}(V)$  to  $\mathcal{P}(V')$ , i.e. for every  $h \in \text{PGL}(V)$  there is  $h' \in \text{PGL}(V')$  such that  $fh = h'f$ . Denote by  $V_f$  the minimal subspace of  $V'$  containing all elements of  $f(\mathcal{P}(V))$ . Our first result is the following: if  $V$  is finite-dimensional and  $\dim V_f \leq \dim V$  then  $f$  is induced by a strong semilinear embedding. The second result states that  $f^{-1}$  is a semicollineation if  $f$  is bijective. As a consequence, we get a characterization of collineations in terms of the projective linear group. Our third result is a characterization of semilinear homeomorphisms of normed spaces.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

Let  $V$  and  $V'$  be left vector spaces over division rings  $R$  and  $R'$ , respectively. Throughout the paper we suppose that the dimensions of  $V$  and  $V'$  both are not less than 3 (possibly the vector spaces are infinite-dimensional). Denote by  $\mathcal{P}(V)$  the projective space associated to  $V$  (the points are 1-dimensional subspaces of  $V$  and the lines are defined by 2-dimensional subspaces of  $V$ , i.e. the line corresponding to a 2-dimensional subspace  $S$  consists of all 1-dimensional subspaces contained in  $S$ ). Similarly, we write  $\mathcal{P}(V')$  for the projective space associated to  $V'$ .

A mapping  $l : V \rightarrow V'$  is called *semilinear* if

$$l(x + y) = l(x) + l(y) \quad \forall x, y \in V$$

and there is a homomorphism  $\sigma : R \rightarrow R'$  such that

$$l(ax) = \sigma(a)l(x) \quad \forall a \in R, x \in V.$$

If  $R = R'$  and  $\sigma$  is identity then the mapping  $l$  is linear.

A semilinear bijection  $l : V \rightarrow V'$  is called a *semilinear isomorphism* if the associated homomorphism of division rings is an isomorphism (the fact that  $l$  is bijective does not guarantee that the associated homomorphism of division rings is an isomorphism). We say that a semilinear injection is a *strong semilinear embedding* if it transfers any collection of linearly independent vectors to a collection of linearly independent vectors.

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Every semilinear injection  $l : V \rightarrow V'$  induces the mapping

$$\pi(l) : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$$

which transfers every 1-dimensional subspace  $P \subset V$  to the 1-dimensional subspace of  $V'$  containing  $l(P)$ . For every non-zero scalar  $a \in R'$  we have  $\pi(al) = \pi(l)$ . Conversely, if  $l(V)$  contains two linearly independent vectors and  $s : V \rightarrow V'$  is a semilinear injection such that  $\pi(l) = \pi(s)$  then  $s$  is a scalar multiple of  $l$ . This means that the projective linear group  $\text{PGL}(V)$  can be identified with the group of all transformations of  $\mathcal{P}(V)$  induced by linear automorphisms of  $V$ , i.e. the transformations of type  $\pi(u)$ , where  $u \in \text{GL}(V)$ .

For every strong semilinear embedding  $l : V \rightarrow V'$  the mapping  $f = \pi(l)$  is an injection satisfying the following condition:

(PGL) for every  $h \in \text{PGL}(V)$  there is  $h' \in \text{PGL}(V')$  such that  $fh = h'f$ .

This holds also if  $f$  is a constant mapping. Every mapping  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  satisfying the above condition is called a *PGL-mapping*. There exist non-constant PGL-mappings which are not induced by semilinear mappings [8, Example 6].

We will use the following notation: for every mapping  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  denote by  $V_f$  the minimal subspace of  $V'$  containing all elements of  $f(\mathcal{P}(V))$ .

**Theorem 1.** *If  $V$  is finite-dimensional then every non-constant PGL-mapping  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  satisfying*

$$\dim V_f \leq \dim V$$

*is induced by a strong semilinear embedding of  $V$  in  $V'$ .*

In [8] this statement is proved under the assumption that  $R'$  is a field. The proof is based on the description of so-called PGL-subsets of  $\mathcal{P}(V)$  (see Section 2 for the definition) in the case when  $R$  is a field. We are not able to classify PGL-subsets in the non-commutative case, but we establish one property of PGL-subsets sufficient to prove the statement in the general (not necessarily commutative) case.

The next two results concern bijective PGL-mappings.

Three distinct points in a projective space are called *collinear* if there is a line containing them; otherwise, these points are said to be *non-collinear*. A bijection between two projective spaces is a *semicollineation* if it transfers any triple of collinear points to a triple of collinear points (in this case, lines go to subsets of lines). A semicollineation is a *collineation* if the inverse mapping also is a semicollineation, or equivalently, three points are collinear if and only if their images are collinear. By the Fundamental Theorem of Projective Geometry, every collineation of  $\mathcal{P}(V)$  to  $\mathcal{P}(V')$  is induced by a semilinear isomorphism of  $V$  to  $V'$ . In the case when  $\dim V \leq \dim V' < \infty$ , every semicollineation of  $\mathcal{P}(V)$  to  $\mathcal{P}(V')$  is a collineation. There is a semicollineation of the projective space associated to a 5-dimensional vector space to a non-desarguesian projective plane [2]. The following problem is still open: are there semicollineations of  $\mathcal{P}(V)$  to  $\mathcal{P}(V')$  which are not collineations? Every semicollineation of  $\mathcal{P}(V)$  to  $\mathcal{P}(V')$  is induced by a semilinear injection of  $V$  to  $V'$  (this follows, for example, from Theorem 4). Thus the above mentioned problem can be reformulated in terms of semilinear mappings.

**Theorem 2.** *If  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  is a bijective PGL-mapping then  $f^{-1}$  is a semicollineation.*

**Corollary 1.** *For every bijection  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  the following two conditions are equivalent:*

- (1)  $f$  is a collineation,
- (2)  $f$  and  $f^{-1}$  both are PGL-mappings.

The implication (1)  $\implies$  (2) follows from the Fundamental Theorem of Projective Geometry. The implication (2)  $\implies$  (1) is a consequence of Theorem 2.

**Remark 1.** Every non-constant PGL-mapping  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  induces a monomorphism of  $\text{PGL}(V)$  to  $\text{PGL}(V_f)$  (see Lemma 2) which is an isomorphism if  $f$  is bijective and  $f^{-1}$  also is a PGL-mapping. All automorphisms of the projective linear group over a finite-dimensional vector space are known [3, Section IV.6]. Automorphisms and isomorphisms of the projective linear groups of infinite-dimensional vector spaces are not described.

Consider a normed space  $\mathcal{N} = (N, |\cdot|)$ , where  $N$  is a real or complex vector space and  $|\cdot|$  is a norm. The group of all linear automorphisms of  $\mathcal{N}$  will be denoted by  $\text{GL}(\mathcal{N})$ . In the case when  $N$  is infinite-dimensional, this is a proper subgroup of the group  $\text{GL}(N)$  formed by linear automorphisms of  $N$ . We write  $\text{PGL}(\mathcal{N})$  for the group of all transformations of  $\mathcal{P}(N)$  induced by elements of  $\text{GL}(\mathcal{N})$ , i.e.  $\text{PGL}(\mathcal{N})$  consists of all  $\pi(u)$  such that  $u \in \text{GL}(\mathcal{N})$ .

Let  $\mathcal{N}' = (N', |\cdot|')$  be another real or complex normed space. We say that  $f : \mathcal{P}(N) \rightarrow \mathcal{P}(N')$  is a  $\text{PGL}^c$ -mapping if for every  $h \in \text{PGL}(\mathcal{N})$  there exists  $h' \in \text{PGL}(\mathcal{N}')$  such that  $fh = h'f$ .

**Theorem 3.** For every bijection  $f : \mathcal{P}(N) \rightarrow \mathcal{P}(N')$  the following two conditions are equivalent:

- (1)  $f$  is induced by a semilinear homeomorphism of  $\mathcal{N}$  to  $\mathcal{N}'$ ,
- (2)  $f$  and  $f^{-1}$  both are  $\text{PGL}^c$ -mappings.

**Remark 2.** The condition (1) implies that the vector spaces  $N$  and  $N'$  are over the same field. Since every endomorphism of  $\mathbb{R}$  is zero or identity, every semilinear mapping between real vector spaces is linear. If  $N$  and  $N'$  are complex vector spaces then every semilinear homeomorphism of  $\mathcal{N}$  to  $\mathcal{N}'$  is linear or the associated endomorphism of  $\mathbb{C}$  is the complex conjugate mapping.

## 2. PGL-SUBSETS

A subset  $\mathcal{X} \subset \mathcal{P}(V)$  will be called a PGL-subset if every permutation on  $\mathcal{X}$  can be extended to an element of  $\text{PGL}(V)$ .

We say that  $P_1, \dots, P_m \in \mathcal{P}(V)$  form an *independent* subset if non-zero vectors  $x_1 \in P_1, \dots, x_m \in P_m$  are linearly independent. In the case when  $V$  is infinite-dimensional, an infinite subset  $\mathcal{X} \subset \mathcal{P}(V)$  is said to be *independent* if every finite subset of  $\mathcal{X}$  is independent. Every independent subset is a PGL-subset.

Let  $m$  be an integer not less than 2. An  $(m+1)$ -element subset  $\mathcal{X} \subset \mathcal{P}(V)$  is called an *m-simplex* if it is not independent and every  $m$ -element subset of  $\mathcal{X}$  is independent. If  $P_1, \dots, P_{m+1} \in \mathcal{P}(V)$  form an  $m$ -simplex then there exist non-zero vectors  $x_1 \in P_1, \dots, x_m \in P_m$  such that

$$P_{m+1} = \langle x_1 + \dots + x_m \rangle.$$

Every  $m$ -simplex is a PGL-subset [1, Section III.3, Proposition 1].

**Proposition 1.** If  $\mathcal{X}$  is a PGL-subset of  $\mathcal{P}(V)$  (possibly infinite) then one of the following possibilities is realized:

- $\mathcal{X}$  is independent;
- there exists an integer  $m$  ( $m < |\mathcal{X}|$  if  $\mathcal{X}$  is finite) such that every  $m$ -element subset of  $\mathcal{X}$  is independent and all elements of  $\mathcal{X}$  are contained in a certain  $m$ -dimensional subspace of  $V$ .

*Proof.* Suppose that  $\mathcal{X}$  is not independent. Then it contains a finite subset  $\mathcal{X}'$  which is not independent (we have  $\mathcal{X} = \mathcal{X}'$  if  $\mathcal{X}$  is finite). We consider any maximal independent subset of  $\mathcal{X}'$ , denote by  $P_1, \dots, P_m$  the elements of this subset and define  $S := P_1 + \dots + P_m$ .

Let  $P$  be an element of  $\mathcal{X} \setminus \{P_1, \dots, P_m\}$  contained in  $S$  (for example, every element of  $\mathcal{X}' \setminus \{P_1, \dots, P_m\}$  satisfies this condition). Then

$$P = \langle a_1 x_1 + \dots + a_m x_m \rangle,$$

where every  $x_i \in P_i$  is non-zero. Show that every scalar  $a_i$  is non-zero.

Consider the PGL-subset  $\{P_1, \dots, P_m, P\}$  (every subset of a PGL-subset is a PGL-subset). If  $a_i = 0$  and  $a_j \neq 0$  then an element of  $\text{PGL}(V)$  extending the transposition

$$(P_i, P_j) \in S(\{P_1, \dots, P_m, P\})$$

does not leave fixed  $P$  which is impossible.

Now show that every element of  $\mathcal{X}$  is contained in  $S$ . If  $Q \in \mathcal{X}$  is not contained in  $S$  then  $P_1, \dots, P_m, Q$  form an independent subset. Consider the PGL-subset

$$\{P_1, \dots, P_m, P, Q\}.$$

As above, we establish that

$$P = \langle b_1 x_1 + \dots + b_m x_m + b_{m+1} x_{m+1} \rangle,$$

where  $x_{m+1} \in Q \setminus \{0\}$  and every  $b_i$  is non-zero. This contradicts the fact that  $P$  is contained in  $S$ .

Let  $\mathcal{Y}$  be an  $m$ -element subset of  $\mathcal{X}$ . If  $\mathcal{Y}$  is independent then the elements of  $\mathcal{Y}$  span the  $m$ -dimensional subspace. This subspace coincides with  $S$ , since every element of  $\mathcal{X}$  is contained in  $S$ .

In the case when  $\mathcal{Y}$  is not independent, we take any maximal independent subset of  $\mathcal{Y}$  and denote by  $Q_1, \dots, Q_k$  its elements. It is clear that  $k < m$ . Using the above arguments, we show that every element of  $\mathcal{X}$  is contained in  $Q_1 + \dots + Q_k$ . Hence  $S \subset Q_1 + \dots + Q_k$  which contradicts the inequality  $k < m$ .  $\square$

The following lemma is a simple consequence of Proposition 1.

**Lemma 1.** *If a PGL-subset  $\mathcal{X} \subset \mathcal{P}(V)$  contains a  $k$ -element subset which is not independent then every  $k$ -element subset of  $\mathcal{X}$  is not independent.*

### 3. SOME PROPERTIES OF PGL-MAPPINGS

Throughout the section we suppose that  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  is a non-constant PGL-mapping.

**Lemma 2.** *The following assertions are fulfilled:*

- (1) *the mapping  $f$  is injective,*
- (2) *for every  $h \in \text{PGL}(V)$  there exists unique  $\bar{h} \in \text{PGL}(V_f)$  such that  $fh = \bar{h}f$  and the mapping  $h \rightarrow \bar{h}$  is a monomorphism of  $\text{PGL}(V)$  to  $\text{PGL}(V_f)$ .*

*Proof.* Let  $h \in \text{PGL}(V)$ . We take any  $h' \in \text{PGL}(V')$  such that  $fh = h'f$ . Then

$$h'f(\mathcal{P}(V)) = fh(\mathcal{P}(V)) = f(\mathcal{P}(V)),$$

i.e.  $h'$  transfers  $f(\mathcal{P}(V))$  to itself. Since  $V_f$  is the minimal subspace of  $V'$  containing all elements of  $f(\mathcal{P}(V))$ , we get  $h'(\mathcal{P}(V_f)) = \mathcal{P}(V_f)$ . If  $h'' \in \text{PGL}(V')$  and  $fh = h''f$  then for every  $P' = f(P)$ ,  $P \in \mathcal{P}(V)$  we have

$$h'(P') = h'f(P) = fh(P) = h''f(P) = h''(P').$$

Thus  $h'|_{f(\mathcal{P}(V))} = h''|_{f(\mathcal{P}(V))}$  which implies that  $h'|_{V_f} = h''|_{V_f}$ .

So, for every  $h \in \text{PGL}(V)$  there is unique  $\bar{h} \in \text{PGL}(V_f)$  satisfying  $fh = \bar{h}f$ .

(1). Suppose that  $f(P) = f(Q)$  for some distinct  $P, Q \in \mathcal{P}(V)$ . Then

$$fh(P) = \bar{h}f(P) = \bar{h}f(Q) = fh(Q) \quad \forall h \in \text{PGL}(V).$$

For every  $T \in \mathcal{P}(V) \setminus \{P\}$  there exists  $h \in \text{PGL}(V)$  leaving fixed  $P$  and transferring  $Q$  to  $T$ . Then

$$f(P) = fh(P) = fh(Q) = f(T)$$

and  $f$  is constant which contradicts our assumption.

(2). If  $h_1, h_2 \in \text{PGL}(V)$  then

$$\overline{h_1 h_2} f = f h_1 h_2 = \bar{h}_1 f h_2 = \bar{h}_1 \bar{h}_2 f.$$

By the above arguments,  $\bar{h}_1 \bar{h}_2 = \overline{h_1 h_2}$ , i.e.  $h \rightarrow \bar{h}$  is a homomorphism of  $\text{PGL}(V)$  to  $\text{PGL}(V_f)$ . Show that this is a monomorphism. If  $\bar{h}$  is the identity element of  $\text{PGL}(V_f)$  then

$$fh(P) = \bar{h}f(P) = f(P) \quad \forall P \in \mathcal{P}(V).$$

Since  $f$  is injective,  $h(P) = P$  for every  $P \in \mathcal{P}(V)$ . □

Let  $S$  be a subspace of  $V$ . Denote by  $S_f$  the minimal subspace of  $V'$  containing all elements of  $f(\mathcal{P}(S))$ . We say that  $S$  is *invariant* for  $h \in \text{PGL}(V)$  if  $h$  transfers  $\mathcal{P}(S)$  to itself, in other words,  $S$  is invariant for every  $u \in \text{GL}(V)$  satisfying  $h = \pi(u)$ .

**Lemma 3.** *For every subspace  $S \subset V$  and every  $h \in \text{PGL}(V)$  the following assertions are fulfilled:*

- (1) *if  $S$  is invariant for  $h$  then  $S_f$  is invariant for  $\bar{h}$ ,*
- (2) *if  $h|_{\mathcal{P}(S)}$  is identity then  $\bar{h}|_{\mathcal{P}(S_f)}$  is identity.*

*Proof.* (1). If  $h(\mathcal{P}(S)) = \mathcal{P}(S)$  then

$$\bar{h}f(\mathcal{P}(S)) = fh(\mathcal{P}(S)) = f(\mathcal{P}(S)),$$

i.e.  $\bar{h}$  transfers  $f(\mathcal{P}(S))$  to itself. This implies that  $S_f$  is invariant for  $\bar{h}$ .

(2). If  $h(P) = P$  for a certain  $P \in \mathcal{P}(V)$  then  $\bar{h}f(P) = fh(P) = f(P)$  and we get the claim. □

**Lemma 4.** *The mapping  $f$  transfers PGL-subsets of  $\mathcal{P}(V)$  to PGL-subsets of  $\mathcal{P}(V_f)$ .*

*Proof.* Let  $\mathcal{X}$  be a PGL-subset of  $\mathcal{P}(V)$ . By the first part of Lemma 2,  $f|_{\mathcal{X}}$  is a bijection to  $f(\mathcal{X})$ . For every permutation  $s'$  on  $f(\mathcal{X})$  there exists a permutation  $s$  on  $\mathcal{X}$  satisfying  $fs = s'f|_{\mathcal{X}}$ . Suppose that  $h \in \text{PGL}(V)$  is an extension of  $s$ . If  $P' \in f(\mathcal{X})$  then  $P' = f(P)$  for a certain  $P \in \mathcal{X}$  and

$$\bar{h}(P') = \bar{h}f(P) = fh(P) = fs(P) = s'f(P) = s'(P').$$

Therefore,  $\bar{h}|_{f(\mathcal{X})} = s'$  and  $\bar{h}$  is an extension of  $s'$ . □

## 4. PROOF OF THEOREM 1

In this section we suppose that  $\dim V = n$  is finite and  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  is a non-constant PGL-mapping satisfying  $\dim V_f \leq n$ .

**Lemma 5.** *The mapping  $f$  transfers every independent subset consisting of  $n - 1$  elements to an independent subset.*

*Proof.* Let  $\mathcal{X}$  be an independent subset of  $\mathcal{P}(V)$  consisting of  $n - 1$  elements. Denote by  $S$  the  $(n - 1)$ -dimensional subspace of  $V$  containing all elements of  $\mathcal{X}$ . It follows from Lemma 4 that  $f(\mathcal{X})$  is a PGL-subset of  $\mathcal{P}(V_f)$ . Suppose that  $f(\mathcal{X})$  is not independent. Then, by Proposition 1, there exists an integer  $m < n - 1$  such that every  $m$ -element subset of  $f(\mathcal{X})$  is independent and all elements of  $f(\mathcal{X})$  are contained in a certain  $m$ -dimensional subspace  $S' \subset V'$ .

Consider  $P \in \mathcal{P}(V)$  which is not contained in  $S$ . We take any  $m$ -element subset  $\mathcal{Y} \subset \mathcal{X}$  and  $Q \in \mathcal{X} \setminus \mathcal{Y}$ . Then  $\mathcal{Y} \cup \{Q, P\}$  is an independent subset and, by Lemma 4,

$$f(\mathcal{Y}) \cup \{f(Q), f(P)\}$$

is a PGL-subset of  $\mathcal{P}(V_f)$ . The subset  $f(\mathcal{Y}) \cup \{f(Q)\}$  is not independent and Lemma 1 implies that  $f(\mathcal{Y}) \cup \{f(P)\}$  is not independent. Since  $f(\mathcal{Y})$  is an independent subset whose elements are contained in  $S'$ , we have  $f(P) \subset S'$ .

Now suppose that  $P \in \mathcal{P}(V)$  is contained in  $S$ . Since the intersection of all  $m$ -dimensional subspaces of  $S$  spanned by  $m$ -element subsets of  $\mathcal{X}$  is zero, there exists an  $m$ -element subset  $\mathcal{Z} \subset \mathcal{X}$  such that  $\mathcal{Z} \cup \{P\}$  is independent. We take any  $T \in \mathcal{P}(V)$  which is not contained in  $S$ . Then  $\mathcal{Z} \cup \{T, P\}$  is independent and

$$f(\mathcal{Z}) \cup \{f(T), f(P)\}$$

is a PGL-subset of  $\mathcal{P}(V_f)$  (Lemma 4). It follows from our assumption that  $f(\mathcal{Z})$  is an independent subset consisting of  $m$  elements contained in  $S'$  and it was established above that  $f(T)$  is contained in  $S'$ . Then  $f(\mathcal{Z}) \cup \{f(T)\}$  is not independent and, by Lemma 1,  $f(\mathcal{Z}) \cup \{f(P)\}$  is not independent. This means that  $f(P) \subset S'$ .

So, all elements of  $f(\mathcal{P}(V))$  are contained in the subspace  $S'$  which implies that  $V_f = S' = S_f$ . The second part of Lemma 3 implies that  $\bar{h}$  is the identity element of  $\text{PGL}(V_f)$  for every  $h \in \text{PGL}(V)$  such that  $h|_{\mathcal{P}(S)}$  is identity. This contradicts the second part of Lemma 2.  $\square$

**Remark 3.** In the case when  $R'$  is a field, Lemma 5 follows from Lemma 4 and the classification of PGL-subsets, see [8] for the details.

**Lemma 6.**  *$\dim V_f = n$  and for every  $(n - 1)$ -dimensional subspace  $S \subset V$  the subspace  $S_f$  is  $(n - 1)$ -dimensional.*

*Proof.* Let  $S$  be an  $(n - 1)$ -dimensional subspace of  $V$ . It follows from Lemma 5 that  $\dim S_f \geq n - 1$ . If  $\dim S_f > n - 1$  then the condition  $\dim V_f \leq n$  guarantees that  $S_f = V_f$ . As in the proof of Lemma 5, we show that the latter equality is impossible. So,  $\dim S_f = n - 1$ . If  $\dim V_f < n$  then we have  $S_f = V_f$  again.  $\square$

**Lemma 7.** *For every non-zero subspace  $S \subset V$  we have  $\dim S_f = \dim S$ .*

*Proof.* In the case when  $n = 3$ , the statement follows directly from Lemma 6. Suppose that  $n \geq 4$  and consider any  $(n - 1)$ -dimensional subspace  $U \subset V$ . If  $U$  is invariant for  $h \in \text{PGL}(V)$  then  $U_f$  is invariant for  $\bar{h}$  (the first part of Lemma 3). This means that  $f|_{\mathcal{P}(U)}$  is a PGL-mapping to  $\mathcal{P}(U_f)$ . Lemma 6 implies that

$\dim U_f = n - 1$ . We apply the above arguments to  $f|_{\mathcal{P}(U)}$  and establish that  $\dim S_f = n - 2$  for every  $(n - 2)$ -dimensional subspace  $S \subset U$ . Step by step, we show that  $\dim S_f = \dim S$  for every non-zero subspace  $S \subset V$ .  $\square$

To complete the proof we need the following generalization of the Fundamental Theorem of Projective Geometry.

**Theorem 4** (C. A. Faure, A. Frölicher [4] and H. Havlicek [5]). *If an injective mapping  $g : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  transfers lines to subsets of lines and  $g(\mathcal{P}(V))$  is not contained in a line then it is induced by a semilinear injection of  $V$  to  $V'$ .*

For every 2-dimensional subspace  $S \subset V$  the mapping  $f$  transfers the line defined by  $S$  to a subset in the line corresponding to the 2-dimensional subspace  $S_f$ . The condition  $\dim V_f = n \geq 3$  guarantees that  $f(\mathcal{P}(V))$  is not contained in a line. Therefore,  $f$  is induced by a semilinear injection  $l : V \rightarrow V'$ . Since  $\dim V_f = n$ , the subspace of  $V'$  spanned by  $l(V)$  is  $n$ -dimensional which is possible only in the case when  $l$  is a strong semilinear embedding of  $V$  in  $V'$ .

## 5. PROOF OF THEOREM 2

Let  $f : \mathcal{P}(V) \rightarrow \mathcal{P}(V')$  be a bijective PGL-mapping.

It is clear that three distinct points of  $\mathcal{P}(V)$  are non-collinear if and only if they form an independent subset. Theorem 2 is a simple consequence of the following modification of Lemma 5.

**Lemma 8.** *The mapping  $f$  transfers any triple of non-collinear points to a triple of non-collinear points.*

Indeed, if  $P'_1, P'_2, P'_3 \in \mathcal{P}(V')$  and

$$f^{-1}(P'_1), f^{-1}(P'_2), f^{-1}(P'_3)$$

are non-collinear then, by Lemma 8,  $P_1, P'_2, P'_3$  are non-collinear. Thus  $f^{-1}$  transfer any triple of collinear points to a triple of collinear points, i.e. it is a semicollineations.

*Proof of Lemma 8.* Let  $P_1, P_2, P_3$  be a triple of non-collinear points of  $\mathcal{P}(V)$ . Denote by  $S$  the 3-dimensional subspace of  $V$  containing each  $P_i$ . Suppose that  $f(P_1), f(P_2), f(P_3)$  are collinear points of  $\mathcal{P}(V')$ , i.e. there exists a 2-dimensional subspace  $S' \subset V'$  containing every  $f(P_i)$ .

If the dimension of  $V$  is not less than 4 then  $S$  is a proper subspace of  $V$ . As in the proof of Lemma 5, we show that the inclusion  $f(P) \subset S'$  holds for every  $P \in \mathcal{P}(V)$  which is not contained in  $S$ . Next we establish the same for every  $P \in \mathcal{P}(V)$  contained in  $S$ . So, every element of  $f(\mathcal{P}(V))$  is contained in  $S'$  which is impossible, since  $S'$  is a proper subspace of  $V'$  and  $f$  is bijective.

Let  $\dim V = 3$ . We distinguish the following two cases:

- (1) Every line of  $\mathcal{P}(V)$  contains precisely 3 points. Then  $\mathcal{P}(V)$  is the Fano plane and the same holds for  $\mathcal{P}(V') = f(\mathcal{P}(V))$ .
- (2) Every line of  $\mathcal{P}(V)$  contains more than 3 points.

Let  $P \in \mathcal{P}(V) \setminus \{P_1, P_2, P_3\}$ . If  $P_1, P_2, P_3, P$  form a 3-simplex then

$$\{f(P_1), f(P_2), f(P_3), f(P)\}$$

is a PGL-subset (this follows from Lemma 4, since every 3-simplex is a PGL-subset). Each  $f(P_i)$  is contained in  $S'$  and Lemma 1 guarantees that  $f(P) \subset S'$ . In the case (1), this means that  $f$  is not bijective and we get a contradiction.

Consider the case (2). It was established above that  $f(P) \subset S'$  if  $P_1, P_2, P_3, P$  form a 3-simplex. Suppose that  $P \subset P_1 + P_2$  (the case when  $P$  is contained in  $P_1 + P_3$  or  $P_2 + P_3$  is similar). We choose  $Q \in \mathcal{P}(V)$  such that

$$\{P_1, P_2, P_3, Q\} \text{ and } \{P, P_2, P_3, Q\}$$

are 3-simplices. By the arguments given above, we have  $f(Q) \subset S'$ . Since

$$\{f(P), f(P_2), f(P_3), f(Q)\}$$

is a PGL-subset (Lemma 4) and  $f(Q), f(P_2), f(P_3)$  are contained in  $S'$ , Lemma 1 implies that  $f(P) \subset S'$ . Therefore, every element of  $f(\mathcal{P}(V))$  is contained in  $S'$  which is impossible.  $\square$

## 6. PROOF OF THEOREM 3

The implication (1)  $\implies$  (2) is obvious and we need to prove the converse implication. Suppose that  $f : \mathcal{P}(N) \rightarrow \mathcal{P}(N')$  is a non-trivial  $\text{PGL}^c$ -mapping. Then  $N_f$  is the minimal subspace of  $N'$  containing all elements of  $f(\mathcal{P}(N))$  and  $\mathcal{N}_f := (N_f, |\cdot|')$  is a normed space.

**Lemma 9.** *The following assertions are fulfilled:*

- (1) *the mapping  $f$  is injective,*
- (2) *for every  $h \in \text{PGL}(N)$  there exists unique  $\bar{h} \in \text{PGL}(\mathcal{N}_f)$  such that  $fh = \bar{h}f$  and the mapping  $h \rightarrow \bar{h}$  is a monomorphism of  $\text{PGL}(N)$  to  $\text{PGL}(\mathcal{N}_f)$ .*

*Proof.* Similar to the proof of Lemma 2.  $\square$

**Lemma 10.** *If  $\mathcal{X}$  is a finite PGL-subset of  $\mathcal{P}(N)$  then every permutation on  $\mathcal{X}$  can be extended to an element of  $\text{PGL}(N)$ .*

*Proof.* Let  $S$  be the minimal subspace of  $N$  containing all elements of  $\mathcal{X}$ . Since  $\mathcal{X}$  is finite,  $S$  is finite-dimensional. If  $h \in \text{PGL}(N)$  is an extension of a permutation on  $\mathcal{X}$  then  $S$  is invariant for  $h$ . Thus every permutation on  $\mathcal{X}$  can be extended to an element of  $\text{PGL}(S)$ . Since  $S$  is finite-dimensional, every element of  $\text{PGL}(S)$  is extendable to an element of  $\text{PGL}(N)$ .  $\square$

Using Lemma 10 and the arguments from the proof of Lemma 4, we establish the following.

**Lemma 11.** *The mapping  $f$  transfers every finite PGL-subsets of  $\mathcal{P}(N)$  to a finite PGL-subset of  $\mathcal{P}(N_f)$ .*

**Lemma 12.** *If  $f$  is bijective then  $f^{-1}$  is a semicollineation.*

*Proof.* Similar to the proof of Lemma 8.  $\square$

**Lemma 13.** *Let  $l : N \rightarrow N'$  be a semilinear isomorphism such that  $l$  and  $l^{-1}$  both transfer closed subspaces of codimension 1 to closed subspaces of codimension 1. Then  $l$  is a homeomorphism of  $\mathcal{N}$  to  $\mathcal{N}'$ .*

*Proof.* See [7, Lemma B] and [6, Lemma 2] for the real and complex case, respectively.  $\square$



Now suppose that  $f$  is bijective and  $f^{-1}$  also is a  $\mathrm{PGL}^c$ -mapping. It follows from Lemma 12 that  $f$  is a collineation of  $\mathcal{P}(N)$  to  $\mathcal{P}(N')$  and the Fundamental Theorem of Projective Geometry implies the existence of a semilinear isomorphism  $l : N \rightarrow N'$  such that  $f = \pi(l)$ .

Let  $S$  be a closed subspace of  $\mathcal{N}$  whose codimension is equal to 1. Then  $S_f = l(S)$  is a subspace of  $N'$  and its codimension is equal to 1. Suppose that  $S_f$  is not closed. Then the closure  $\overline{S_f}$  coincides with  $N'$ . We take any non-identity  $h \in \mathrm{PGL}(\mathcal{N})$  such that  $h|_{\mathcal{P}(S)}$  is identity. By the second part of Lemma 3,  $\overline{h}|_{\mathcal{P}(S_f)}$  is identity. Since  $\overline{h}$  belongs to  $\mathrm{PGL}(\mathcal{N}')$ , the restriction of  $\overline{h}$  to  $\overline{S_f}$  also is identity. The equality  $\overline{S_f} = N'$  implies that  $\overline{h}$  is the identity element of  $\mathrm{PGL}(\mathcal{N}')$  which contradicts the second part of Lemma 9.

So,  $l$  transfers closed subspaces of codimension 1 to closed subspaces of codimension 1. The same holds for  $l^{-1}$  (since  $f^{-1}$  is a  $\mathrm{PGL}^c$ -mapping). Lemma 13 gives the claim.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF WARMIA AND MAZURY,  
 SŁONECZNA 54, 10-710 OLSZTYN, POLAND  
*E-mail address:* `pankov@matman.uwm.edu.pl`